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RECURRENCE FORMULAS AND DIFFERENTIAL EQUATIONS

FOR STRESS ANALYSIS OF CAMBERED BOX BEAMS

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SUMMARY

Recurrence formulas and differential equations are developed
for the stress analysis of cambered box beams having one or two covers.
In contrast to the elementary theory of bending, shear deformation of
cover and web is considered. The recurrence formulas are applicable
to beams loaded by concentrated forces at intervals along their spans;
the differential equation is valid for beams of constant section
loaded at the tip and is solved for beams fixed at the root.

For various methods of end restraint, boundary equations are
developed which, together with the pertinent recurrence formula,
yield a set of simultaneous linear algebraic equations. Solution of
these equations provides correction forces which can be used in
conjunction with the results of the elementary bending theory to
determine the direct stresses in the stringers and the shear stresses
in the skin.

Comparison of the results of the application of the formulas
and differential equations developed with experiment and other methods
of analysis indicate that the analysis presented is quite adequate for
the determination of the stresses in cambered box beams.

INTRODUCTION

In the past, satisfactory solutions have been obtained for the
stress analysis of rectangular or flat box beams under bending loads.
(See, for example, references 1 and 2.) Procedures presented for the
analysis of cambered box beams, however, have been for the most part
quite approximate. In the present paper, recurrence formulas are
developed for the stress analysis of single-covered cambered box
beams having singly symmetrical cross sections and double-covered
cambered box beams having doubly symmetrical cross sections. The
recurrence formulas, together with the boundary equations developed,

are applicable to beams loaded symmetrically by concentrated forces
at sections along their spans. The beams can be restrained at both
ends, unrestrained at both ends, or restrained at one end and
unrestrained at the other end. In any particular problem, a
recurrence formula, together with the pertinent boundary equations,
is used to obtain a set of simultaneous linear algebraic equations
for corrections to the stresses given by the elementary theory of
bending. For tip-loaded cantilevered beams of uniform construction,
exact differential-equation solutions are developed.

Applications of both the recurrence formulas and differential
equations and two numerical examples are given. Comparison of the
results of the present analyses with experimental data and other
methods indicates that the solutions presented herein are satisfactory.

SYMBOLS

A_F, A_S, A_L effective cross-sectional area of stringers F, S, and L, respectively

E Young's modulus

G shear modulus

I moment of inertia of cross section with respect to neutral axis

L length of beam

L_i length of bay i

M bending moment on cross section

P load on web

Q static moment about neutral axis of cross-sectional area lying between extreme fiber and plane under consideration

V resultant vertical shear force on cross section

X correction force

b half width of cover measured along cross section

c magnitude of camber

h   depth of web

i   general designation of section or bay

k   fixity factor for carry-through bay

m   designation of bay adjacent to rigid support

t   skin thickness

u, v   axial and chordwise displacements, respectively

x, y   axial and chordwise coordinate axes

y'   cover coordinate axis (See fig. 3(a).)

z   distance between neutral axis and stringer F

δ stringer displacement due to elementary shear stress

$\bar{\delta}(x)$ stringer displacement at distance x due to
             correction shear stress

$$\lambda = \left(1 + \frac{c^2 t_c}{bht_w}\right) \frac{Kb}{Gt_c \tanh KL}$$

$$\mu = \left(1 + \frac{c^2 t_c}{bht_w}\right) \frac{Kb}{Gt_c \sinh KL}$$

$$\nu = \frac{1}{IG} \left\{ A_S(z + c) \frac{b}{t_c} - \left[A_S(z + c) + A_F z\right] \frac{c}{t_w} \right\}$$

$$\lambda' = \left(1 + \frac{2c^2 t_c}{bht_w}\right) \frac{K'b}{Gt_c \tanh K'L}$$

$$\mu' = \left(1 + \frac{2c^2 t_c}{bht_w}\right) \frac{K'b}{Gt_c \sinh K'L}$$

$$\nu' = \frac{1}{2IG} \left\{ A_S(h + 2c) \frac{b}{t_c} - \left[A_S(h + 2c) + A_F h\right] \frac{c}{t_w} \right\}$$

σ total stringer stress

$\bar{\sigma}$ correction stringer stress

τ total shear stress

$\bar{\tau}$ correction shear stress

τ' elementary shear stress

$$K^2 = \frac{\dfrac{Gt_c}{Eb}\left[\dfrac{1}{A_F}\left(\dfrac{h+c}{h}\right)^2 + \dfrac{1}{A_S} + \dfrac{c^2}{h^2 A_L}\right]}{\left(1 + \dfrac{c^2 t_c}{bht_w}\right)}$$

$$K'^2 = \frac{\dfrac{Gt_c}{Eb}\left[\dfrac{1}{A_F}\left(\dfrac{h+2c}{h}\right)^2 + \dfrac{1}{A_S}\right]}{\left(1 + \dfrac{2c^2 t_c}{bht_w}\right)}$$

$$K_1^2 = \frac{\dfrac{G}{Ebh}\left[\dfrac{1}{A_F} + \dfrac{c}{h}\left(\dfrac{1}{A_F} + \dfrac{1}{A_L}\right)\right]}{\left(1 + \dfrac{c^2 t_c}{bht_w}\right)}$$

$$K_2^2 = \frac{\dfrac{c}{bht_w}}{\left(1 + \dfrac{c^2 t_c}{bht_w}\right)}$$

$$K_1'^2 = \frac{\dfrac{G}{EbA_F}\left(\dfrac{h+2c}{h^2}\right)}{\left(1 + \dfrac{2c^2 t_c}{bht_w}\right)}$$

$$K_2'^2 = \frac{\dfrac{c}{bht_w}}{\left(1 + \dfrac{2c^2 t_c}{bht_w}\right)}$$

Subscripts

c   cover

1   bay 1

m   bay m

w   web

F   refers to stringer F

S   refers to stringer S

L   refers to stringer L

INADEQUACY OF ELEMENTARY THEORY

The elementary theory of bending yields for the direct stress and shear stress in a cambered box beam (figs. 1(a) and 1(b)) My/I and VQ/It, respectively. Although the stresses obtained with these relationships satisfy the laws of statics, the corresponding strains do not satisfy continuity conditions. If, for example, a beam is loaded at several sections along its span, as in figure 2(a), the shear strains calculated from the elementary theory permit discontinuities in the box cover at sections at which steps in the shear diagram occur. The discontinuities at each of these sections can be resolved into two components - that due to shear in the cover and that due to shear in the web. As indicated in figure 2(b), the elementary shear strains in the cover tend to displace the upper

stringer toward the supported section. If no shear existed in the cover, however, elements of the cover that were originally perpendicular to the web at the flange (stringer F) would tend to remain so during shear displacement of the web. As shown in figure 2(c), the upper stringer now has the tendency to be displaced away from the supported end. Since the magnitudes of these displacements are proportional to the external shear acting on each bay, the elementary theory permits discontinuities to exist at those sections at which external shear loads are introduced. Discontinuities between adjacent bays having differences in construction are also permitted.

For a tip-loaded beam fixed at points indicated by the circles in figures 2(b) and 2(c), the stresses in the sheet and stringers would be in agreement with elementary theory only if the upper stringer at the supported end is permitted to be displaced in a manner consistent with elementary shear stresses. For beams fixed rigidly at one end, this condition is completely neglected by the elementary theory. It should be noted that, for bending of the beam (without shear deformation), sections that were originally plane remain so after bending occurs and merely rotate with respect to each other. As indicated in figure 2(d), no discontinuities arise when these rotations are permitted.

Because the stresses in box beams determined from the elementary theory do not satisfy conditions of continuity, and consequently are inadequate for the analysis of beams similar to those used in airplane construction, recurrence formulas and differential equations are developed in the present paper for the determination of self-equilibrating correction forces which, when combined with the elementary forces, yield stresses that satisfy the laws of statics as well as the conditions of continuity.

BASIC ASSUMPTIONS OF PRESENT THEORY

In the development of the recurrence formulas for the corrections to the elementary theory, the following basic assumptions are made:

(1) The beam is prismatical and is composed of bays, the cross sections of which have one or two axes of symmetry as shown in figures 1(a) and 1(b).

(2) Each box cover approximates two flat plates joined at a central cover stringer (fig. 1). If the cover is curved, the arc of the cross section of the cover can be replaced by two equal chords.

(3) The skin of covers and webs is capable of carrying only shear stresses; whereas stringers plus an effective sheet area carry only direct stress.

(4) Chordwise displacements are negligible.

DIFFERENTIAL EQUATIONS FOR THE STRESS ANALYSIS

OF CAMBERED BOX BEAMS

Beams with One Cover

A second-order differential equation is found for the stress analysis of single-covered cambered box beams. This equation is applicable to laterally loaded beams having external shear forces symmetrically distributed between the webs. In addition to the basic assumptions listed previously, for the solution of the differential equation, the beam is assumed to be a tip-loaded cantilever box of uniform construction (fig. 1(a)). Because of symmetry, only half the beam is considered in the analysis (fig. 3). The upper stringer of the half cover has half the area of the original cover stringer.

Expressions for stringer and shear stresses.- The origin of coordinates is located at the root as indicated in figure 3(a). The direct stress in each stringer is

$$\sigma = E \frac{du}{dx} \tag{1}$$

in which u represents axial stringer displacement. The shear stresses in the web and cover, respectively, are

$$\left. \begin{aligned} \tau_w &= G \left(\frac{\partial u_w}{\partial y} + \frac{\partial v_w}{\partial x} \right) \\[2ex] \tau_c &= G \left(\frac{\partial u_c}{\partial y'} + \frac{\partial v_c}{\partial x} \right) \end{aligned} \right\} \tag{2}$$

in which u_w and v_w, respectively, are x- and y-displacements of the web, and u_c and v_c, respectively, are the x- and y'-displacements of the cover. The sign convention for the shear stress is indicated on figure 3(b).

If the axial displacements of stringers F, S, and L
(figs. 3(a) and 3(b)) at any transverse section of the beam are
designated as u_F, v_S, and u_L, respectively, equations (2) become

$$\tau_w = G\left(\frac{u_L - u_F}{h} + \frac{\partial v_w}{\partial x}\right)$$

and

$$\tau_c = G\left(\frac{u_F - u_S}{b} + \frac{c}{b}\frac{\partial v_w}{\partial x}\right) \qquad\qquad (3)$$

since

$$v_c = \frac{c}{b} v_w$$

Elimination of $\dfrac{\partial v_w}{\partial x}$ from equations (3) yields

$$\tau_c - \frac{c}{b}\tau_w = \frac{G}{b}\left(u_F - u_S\right) - \frac{Gc}{bh}\left(u_L - u_F\right) \qquad (4)$$

Equilibrium relationships.- Consideration of the static
equilibrium of the forces on the free-body diagram of figure 4 yields
the following differential equations:

$$\frac{d\sigma_F}{dx} = \frac{\tau_c t_c}{A_F} - \frac{\tau_w t_w}{A_F} \qquad\qquad (5a)$$

$$\frac{d\sigma_S}{dx} = -\frac{\tau_c t_c}{A_S} \qquad\qquad (5b)$$

$$\frac{d\sigma_L}{dx} = \frac{\tau_w t_w}{A_L} \qquad\qquad (5c)$$

If any cross section of the beam is loaded by a shear force V,
then from further equilibrium considerations (fig. 3(b))

$$V = \tau_w t_w h + \tau_c t_c c$$

or

$$\tau_w = \frac{V}{t_w h} - \frac{\tau_c t_c c}{t_w h}$$

$$\left. \right\} \quad (6)$$

Differential equation for shear stress in cover.- In order to obtain an expression for τ_c in terms of the physical properties of the beam and the external loading only, τ_w, u_F, u_S, and u_L are eliminated from equation (4). Differentiation of equation (4) twice and subsequent substitution of equations (1), (5), and (6) into the resulting expression yields the following second-order differential equation for the shear stress in the cover of the beam:

$$\left(D^2 - K^2\right)\tau_c = \left(K_2{}^2 D^2 - K_1{}^2\right) V \qquad (7)$$

in which

$$D^2 = \frac{d^2}{dx^2}$$

$$K^2 = \frac{\frac{G t_c}{E b}\left[\frac{1}{A_F}\left(\frac{h+c}{h}\right)^2 + \frac{1}{A_S} + \frac{c^2}{h^2 A_L}\right]}{\left(1 + \frac{c^2 t_c}{b h t_w}\right)}$$

$$K_1{}^2 = \frac{\frac{G}{E b h}\left[\frac{1}{A_F} + \frac{c}{h}\left(\frac{1}{A_F} + \frac{1}{A_L}\right)\right]}{\left(1 + \frac{c^2 t_c}{b h t_w}\right)}$$

$$K_2{}^2 = \frac{\frac{c}{b h t_w}}{\left(1 + \frac{c^2 t_c}{b h t_w}\right)}$$

Solution of differential equation for a tip-loaded beam.- For a tip-loaded beam (fig. 3(a)), equation (7) reduces to

$$\left(D^2 - K^2\right) \tau_0 = -K_1^2 P \tag{8}$$

in which P is the load at the tip of each web. The general solution of equation (8) is

$$\tau_0 = C_1 \sinh Kx + C_2 \cosh Kx + \frac{K_1^2}{K^2} P \tag{9}$$

in which C_1 and C_2 are constants of integration. At the fixed end, $x = 0$ and $u_F = u_S = u_L = 0$. Hence, from equations (4), (6), and (9),

$$C_2 = -\left(\frac{K_1^2}{K^2} - \frac{c}{bht_w + c^2 t_c}\right) P \tag{10}$$

At the free end, $x = L$ and $\sigma_F = \sigma_S = \sigma_L = 0$. Consequently, from equations (1), (4), and (6), at $x = L$,

$$\frac{d\tau_0}{dx} = 0 \tag{11}$$

and therefore from equations (9) and (11)

$$C_1 = \left(\frac{K_1^2}{K^2} - \frac{c}{bht_w + c^2 t_c}\right) P \tanh KL \tag{12}$$

Shear and stringer stresses.- From equations (6), (9), (10), and (12), the shear stresses in the cover and web, respectively, of a tip-loaded cantilevered beam are

$$\tau_0 = -P\left(\frac{K_1^2}{K^2} - \frac{c}{bht_w + c^2 t_c}\right)\frac{\cosh K(L - x)}{\cosh KL} + \frac{K_1^2}{K^2} P \tag{13a}$$

$$\tau_w = \frac{Pt_c c}{t_w h}\left(\frac{K_1^2}{K^2} - \frac{c}{bht_w + c^2 t_c}\right)\frac{\cosh K(L-x)}{\cosh KL} + \frac{P}{t_w h}\left(1 - ct_c \frac{K_1^2}{K^2}\right)$$

<div align="right">(13b)</div>

The corresponding stresses in stringers F, S, and L, respectively, are from equations (5) and (13)

$$\sigma_F = \frac{Pt_c}{KA_F}\left(\frac{h+c}{h}\right)\left(\frac{K_1^2}{K^2} - \frac{c}{bht_w + c^2 t_c}\right)\frac{\sinh K(L-x)}{\cosh KL}$$

$$+ \frac{P}{A_F h}\left[t_c(h+c)\frac{K_1^2}{K^2} - 1\right](x-L)$$

<div align="right">(14a)</div>

$$\sigma_S = -\frac{Pt_c}{KA_S}\left(\frac{K_1^2}{K^2} - \frac{c}{bht_w + c^2 t_c}\right)\frac{\sinh K(L-x)}{\cosh KL} - \frac{t_c K_1^2}{A_S K^2}P(x-L)$$ (14b)

$$\sigma_L = -\frac{Pct_c}{KA_L h}\left(\frac{K_1^2}{K^2} - \frac{c}{bht_w + c^2 t_c}\right)\frac{\sinh K(L-x)}{\cosh KL}$$

$$+ \frac{P}{A_L h}\left(1 - ct_c \frac{K_1^2}{K^2}\right)(x-L)$$

<div align="right">(14c)</div>

In equations (13) and (14) those terms containing hyperbolic functions represent corrections to the stresses obtained with the elementary theory. The other terms represent the elementary stresses. If a substitute structure is considered, therefore, the correction terms only should be found and added to the elementary stresses that correspond to the actual structure.

Beams with Two Covers

A typical beam having upper and lower covers of equal camber is shown in figure 1(b). Because of the double-symmetry of the cross section, the analysis of this box is somewhat simpler than that for the single-covered box beam. For the double-covered beam, equations (4), (5), and (6), respectively, are replaced by (fig. 5)

$$\tau_c - \frac{c}{b}\tau_w = \frac{G}{b}\left(u_F - u_S\right) + \frac{2Gc}{bh}u_F \qquad (15)$$

$$\frac{d\sigma_F}{dx} = \frac{\tau_c t_c}{A_F} - \frac{\tau_w t_w}{A_F} \qquad (16)$$

$$\frac{d\sigma_S}{dx} = -\frac{\tau_c t_c}{A_S} \qquad (17)$$

$$\tau_w = \frac{V}{t_w h} - \frac{2\tau_c t_c c}{t_w h} \qquad (18)$$

Elimination of τ_w, u_F, and u_S from equations (1), and (15) to (18) yields the following differential equation for the shear stress in either cover of the beam

$$\left(D^2 - K'^2\right)\tau_c = \left(K_2'^2 D^2 - K_1'^2\right)V \qquad (19)$$

in which

$$K'^2 = \frac{\dfrac{Gt_c}{Eb}\left[\dfrac{1}{A_F}\left(\dfrac{h+2c}{h}\right)^2 + \dfrac{1}{A_S}\right]}{\left(1 + \dfrac{2c^2 t_c}{bh t_w}\right)}$$

$$K_1'^2 = \frac{\dfrac{G}{EbA_F}\left(\dfrac{h+2c}{h^2}\right)}{\left(1 + \dfrac{2c^2 t_c}{bh t_w}\right)}$$

$$K_2'^2 = \frac{\dfrac{c}{bh t_w}}{\left(1 + \dfrac{2c^2 t_c}{bh t_w}\right)}$$

For a beam loaded at the tip of each web by a force P, equation (19) becomes

$$\left(D^2 - K'^2\right)\tau_c = -K_1'^2 P \tag{20}$$

Solution of this equation for a beam fixed at $x = 0$ yields the following expressions for the shear stress in the covers and webs, respectively,

$$\left.\begin{aligned}
\tau_c &= -P\left(\frac{K_1'^2}{K'^2} - \frac{c}{bht_w + 2c^2 t_c}\right)\frac{\cosh K'(L - x)}{\cosh K'L} + \frac{K_1'^2}{K'^2} P \\[2em]
\tau_w &= \frac{2Pt_c c}{t_w h}\left(\frac{K_1'^2}{K'^2} - \frac{c}{bht_w + 2c^2 t_c}\right)\frac{\cosh K'(L - x)}{\cosh K'L} + \frac{P}{t_w h}\left(1 - 2ct_c\frac{K_1'^2}{K'^2}\right)
\end{aligned}\right\} \tag{21}$$

The stresses in the stringer of the tension cover of the beam (upper cover in fig. 1(b)) are of equal magnitude but opposite in sign to the corresponding stresses in the stringers of the compression cover and are

$$\left.\begin{aligned}
\sigma_F &= \frac{Pt_c}{K'A_F}\left(\frac{h + 2c}{h}\right)\left(\frac{K_1'^2}{K'^2} - \frac{c}{bht_w + 2c^2 t_c}\right)\frac{\sinh K'(L - x)}{\cosh K'L} \\[1em]
&\quad + \frac{P}{A_F h}\left[t_c(h + 2c)\frac{K_1'^2}{K'^2} - 1\right](x - L) \\[2em]
\sigma_S &= -\frac{Pt_c}{K'A_S}\left(\frac{K_1'^2}{K'^2} - \frac{c}{bht_w + 2c^2 t_c}\right)\frac{\sinh K'(L - x)}{\cosh K'L} - \frac{t_c K_1'^2}{A_S K'^2} P(x - L)
\end{aligned}\right\} \tag{22}$$

In equations (21) and (22) those terms containing hyperbolic functions represent corrections to the corresponding stresses determined by the elementary theory.

DEVELOPMENT OF RECURRENCE FORMULAS AND BOUNDARY EQUATIONS

Recurrence Formula for Single-Covered Box

Procedure.- At any section i of a cambered box beam with a single symmetrical cover (see fig. 6(a)), the discontinuity or "gap" in the cover permitted by the elementary theory is determined. A suitable set of stringer correction forces, which can be related to stringer displacement at any section i, is found and the manner in which these forces distribute themselves within a bay is determined. A recurrence formula is obtained which, when applied to successive sections, yields a set of simultaneous equations for the correction force in the cover stringer at each section considered. The corrections to the elementary stringer and shear stresses are therefore readily determined.

Stringer displacements due to elementary shear stresses.- In order to determine the magnitude of the discontinuities that would exist between adjacent bays on the basis of elementary theory, the stringer displacements at any section i at which changes in external loading or bay construction take place are found from the elementary shear stresses. Because of the symmetry only one half of the beam cross section is considered (fig. 6(b)). The upper stringer of the half cover has half the area of the original cover stringer.

The elementary shear stresses in the cover and web of bay i are, respectively,

$$\tau'_{c_i} = \frac{V_i A_{S_i}}{I_i t_{c_i}} \left(z_i + c \right) \tag{23}$$

and

$$\tau'_{w_i} = \frac{V_i}{I_i t_{w_i}} \left[A_{S_i} \left(z_i + c \right) + A_{F_i} z_i \right] \tag{24}$$

in which

A_F, A_S cross sectional effective area of stringers F and S, respectively

I_i moment of inertia with respect to neutral axis of cross section of bay i

c   camber (vertical distance of stringer S from stringer F)

t_{c_i}, t_{w_i} skin thickness of cover and web, respectively, of bay i

z   distance of stringer F from neutral axis.

The sign convention for the elementary shear stresses is indicated in figure 6(b).

If the part of stringer S within bay i is to remain continuous with the sheet of the cover of bay i, because of the elementary shear stresses, each point of the stringer segment must be displaced a distance (fig 6(c))

$$\delta_i = \tau'_{w_i}\frac{c}{G} - \tau'_{c_i}\frac{b}{G} \tag{25}$$

in which G is the shear modulus and b, the half cover width.

The first term of the right-hand side of equation (25) represents the contribution of the web shear stress to the stringer displacement, whereas the second term is the contribution of the cover shear stress. From equations (23) and (24), equation (25) becomes

$$\delta_i = -\nu_i V_i \tag{26}$$

in which

$$\nu_i = \frac{1}{I_i G}\left\{ A_{S_i}\left(z_i + c\right)\frac{b}{t_{c_i}} - \left[A_{S_i}\left(z_i + c\right) + A_{F_i}z_i\right]\frac{c}{t_{w_i}}\right\}$$

From elementary considerations, therefore, the total gap between the segments of stringer S of adjacent bays i and i - 1 (see fig. 6(c)) is

$$\delta_i - \delta_{i-1} = \nu_{i-1}V_{i-1} - \nu_i V_i \tag{27}$$

Equation (27) indicates that, on the basis of elementary theory, displacements of the cover of adjacent bays having different applied

external shear forces or variations in structural properties
result in discontinuities in the cover.

Correction forces.- At any section i of the beam, disconti-
nuities can be eliminated by the application of a suitable set of
self-equilibrating correction forces to stringers F, S, and L.
If a force $-X_i$ is applied to stringer S (fig. 6(d)) the applied
forces required at stringers F and L are, respectively,

$$\left.\begin{array}{c} \dfrac{h + c}{h} X_i \\[2em] -\dfrac{c}{h} X_i \end{array}\right\} \quad (28)$$

and

where h is the depth of the web. In order to utilize these forces
for the elimination of the discontinuities in the cover, a relation-
ship must be determined between the displacements of stringer S and
the correction force X_i.

Relationships between correction force and stresses.- In order
to find the desired relationship between stringer displacements and
correction force, the shear stresses in the cover and web (see
fig. 6(d)) are expressed as a function of X. Since the set of
correction forces yields no resultant vertical shear, equation (6)
for any section i, reduces to

$$\overline{\tau}_{w_i} = - \frac{\overline{\tau}_{c_i} t_{c_i} c}{t_{w_i} h} \qquad (29)$$

where the horizontal bar indicates correction stresses. From
equations (1), (4), and (29), the shear stress in the cover of bay i
due to the application of the correction forces is expressible as

$$\left(1 + \frac{c^2 t_{c_i}}{bht_{w_i}}\right) \frac{d\overline{\tau}_{c_i}}{dx_i} = \frac{G}{Eb}\left(1 + \frac{c}{h}\right)\overline{\sigma}_{F_i} - \frac{G}{Eb}\overline{\sigma}_{S_i} - \frac{Gc}{Ebh}\overline{\sigma}_{L_i} \qquad (30)$$

in which $\overline{\sigma}_{F_i}$, $\overline{\sigma}_{S_i}$, and $\overline{\sigma}_{L_i}$ are the correction stringer stresses
for stringers F, S, and L, respectively, at any distance x_i from

section i. In figure 6(d) it is seen that at $x_i = 0$,

$$\left. \begin{aligned} \overline{\sigma}_{F_i} &= \frac{h + c}{hA_{F_i}} X_i \\[2mm] \overline{\sigma}_{S_i} &= -\frac{X_i}{A_{S_i}} \\[2mm] \overline{\sigma}_{L_i} &= -\frac{c}{hA_{L_i}} X_i \end{aligned} \right\} \tag{31}$$

and

Substitution of equations (31) into equation (30) yields at $x_i = 0$

$$\left(1 + \frac{c^2 t_{c_i}}{bht_{w_i}}\right)\frac{d\overline{\tau}_{c_i}}{dx_i} = \frac{G}{Eb}\left(1 + \frac{c}{h}\right)\left(\frac{h+c}{h}\right)\frac{X_i}{A_{F_i}} + \frac{G}{Eb}\frac{X_i}{A_{S_i}} + \frac{Gc}{Ebh}\frac{c}{h}\frac{X_i}{A_{L_i}} \tag{32}$$

or at $x_i = 0$

$$\frac{d\overline{\tau}_{c_i}}{dx_i} = \frac{K_i^2}{t_{c_i}} X_i \tag{33}$$

in which

$$K_i^2 = \frac{\dfrac{Gt_{c_i}}{Eb}\left[\dfrac{1}{A_{F_i}}\left(\dfrac{h+c}{h}\right)^2 + \dfrac{1}{A_{S_i}} + \dfrac{1}{A_{L_i}}\dfrac{c^2}{h^2}\right]}{\left(1 + \dfrac{c^2 t_{c_i}}{bht_{w_i}}\right)}$$

At $x_i = L_i$

$$\overline{\sigma}_{F_i} = \overline{\sigma}_{S_i} = \overline{\sigma}_{L_i} = 0$$

and from equation (30), therefore, at $x_i = L_i$,

$$\frac{d\bar{\tau}_{c_i}}{dx_i} = 0 \qquad (34)$$

The manner in which the shear stress in the beam cover is distributed consistent with no resultant shear on a cross section is obtained from equation (7) which reduces to

$$\left(D^2 - K_i{}^2\right)\bar{\tau}_{c_i} = 0 \qquad (35)$$

The general solution of this homogeneous second-order differential equation is

$$\bar{\tau}_{c_i} = C_1 \sinh K_i x_i + C_2 \cosh K_i x_i \qquad (36)$$

Introduction of the boundary conditions represented by equations (33) and (34) into equation (36) yields the following relationship between the shear stress in the cover of bay i and the applied correction force X_i

$$\bar{\tau}_{c_i} = -X_i \frac{K_i}{t_{c_i}} \frac{\cosh K_i\left(L_i - x_i\right)}{\sinh K_i L_i} \qquad (37)$$

From equations (29) and (37) the shear stress in the web is

$$\bar{\tau}_{w_i} = X_i \frac{K_i c}{t_{w_i} h} \frac{\cosh K_i\left(L_i - x_i\right)}{\sinh K_i L_i} \qquad (38)$$

Relationships between correction force and stringer displacements.- Within any bay i the displacement of stringer S corresponding to the correction shear stresses $\bar{\tau}_{c_i}$ and $\bar{\tau}_{w_i}$ and, consequently, the correction force X_i, is determined from equation (25) and is

$$\bar{\delta}_i\left(x_i\right) = \bar{\tau}_{w_i} \frac{c}{G} - \bar{\tau}_{c_i} \frac{b}{G} \qquad (39)$$

Substitution of equations (27) and (28) into equation (39) yields

$$\bar{\delta}_i(x_i) = \left(1 + \frac{c^2 t_{c_i}}{bht_{w_i}}\right) \frac{bK_i}{t_{c_i}G} \frac{\cosh K_i\left(L_i - x_i\right)}{\sinh K_i L_i} X_i \qquad (40)$$

Equation (40) relates the axial displacement of stringer S at any point x_i to the correction force X_i. If bay i has a length L_i, the displacement of the cover stringer at $x_i = 0$ and $x_i = L_i$ are, respectively,

$$\bar{\delta}_i(0) = \lambda_i X_i \qquad (41)$$

in which

$$\lambda_i = \left(1 + \frac{c^2 t_{c_i}}{bht_{w_i}}\right) \frac{K_i b}{Gt_{c_i} \tanh K_i L_i}$$

and

$$\bar{\delta}_i\left(L_i\right) = \mu_i X_i \qquad (42)$$

in which

$$\mu_i = \left(1 + \frac{c^2 t_{c_i}}{bht_{w_i}}\right) \frac{K_i b}{Gt_{c_i} \sinh K_i L_i}$$

Development of recurrence formula.- A recurrence formula can now be obtained from consideration of the amount of correction force X_i required to eliminate the discontinuity determined from the elementary theory. From equation (27) the magnitude of the discontinuity between two adjacent bays i and $i - 1$ (see fig. 6(c)) is

$$\delta_i - \delta_{i-1} = \nu_{i-1}V_{i-1} - \nu_i V_i \qquad (43)$$

Since, in general, correction forces are applied to both ends of a bay, the discontinuity defined by equation (43) is eliminated by the application of correction forces X_{i-1}, X_i, and X_{i+1} (fig. 7). Consequently, from equations (41) and (42) the discontinuity at section i can be eliminated by making

$$\delta_i - \delta_{i-1} = \left(\mu_i X_{i+1} - \lambda_i X_i\right) - \left(\lambda_{i-1} X_i - \mu_{i-1} X_{i-1}\right) \qquad (44)$$

Combination of equations (43) and (44) yields the following recurrence formula relating the correction forces X at three successive sections:

$$\mu_{i-1} X_{i-1} - \left(\lambda_{i-1} + \lambda_i\right) X_i + \mu_i X_{i+1} = \nu_{i-1} V_{i-1} - \nu_i V_i \qquad (45)$$

The recurrence formula (45) relates the correction force applied to the cover stringer S at section i to the corresponding forces applied at the sections adjacent to section i. One equation, therefore, similar to equation (45) can be written for each section of a single-covered cambered box beam provided that at least one bay exists on each side of that section.

Boundary Equations

Equations suitable for application to bays adjacent to boundaries can be readily obtained from the previous analysis and the pertinent boundary conditions. Consequently, boundary equations are presented herein for conditions of full restraint, partial restraint, or no restraint. By suitable combinations of the boundary equations, and proper manipulation of the subscripts, these equations can be applied to the analysis of beams restrained at both ends, unrestrained at both ends, or restrained at one end and unrestrained at the other.

Boundary equation for fixed end.- If a beam is fixed to the right of bay m as indicated in figure 8, the stringer displacements at section $m + 1$ must be zero. Consequently, from equations (41) and (42)

$$\delta_m = \lambda_m X_{m+1} - \mu_m X_m \qquad (46)$$

and from equation (28), therefore, the fixed-end boundary equation
is

$$\mu_m X_m - \lambda_m X_{m+1} = \nu_m V_m \qquad (47)$$

If the beam is fixed to left of bay 0, the boundary equation is

$$\lambda_0 X_0 - \mu_0 X_1 = \nu_0 V_0 \qquad (48)$$

Boundary equation for unrestrained end.- If the beam of figure 8
is unrestrained to the left of bay 0, the correction force at
section 0 is zero. Therefore, from continuity considerations at
section 1 and from equations (27), (41), and (42), the unrestrained-
end boundary equation which is written for section 1 is

$$-\left(\lambda_0 + \lambda_1\right) X_1 + \mu_1 X_2 = \nu_0 V_0 - \nu_1 V_1 \qquad (49)$$

and if bay m is unrestrained at its right, the boundary equation for
bay m is

$$\mu_{m-1} X_{m-1} - \left(\lambda_{m-1} + \lambda_m\right) X_m = \nu_{m-1} V_{m-1} - \nu_m V_m \qquad (50)$$

Boundary equation for partly restrained end.- If a beam is
continuous through a fuselage as shown in figure 9, bay m is
partly restrained at its right end. The boundary equation applicable
to section m + 1 is

$$\mu_m X_m - \left(\lambda_m + k\right) X_{m+1} = \nu_m V_m \qquad (51)$$

in which, when bay m + 1 is composed of both skin and stringers,

$$k = \frac{\lambda^2_{m+1} - \mu^2_{m+1}}{\lambda_{m+1}}$$

and, when bay $m + 1$ is composed of stringers only,

$$k = \frac{L_{m+1}}{hE}\left(\frac{h + c}{A_{F_{m+1}}} + \frac{h}{A_{S_{m+1}}}\right)$$

Recurrence Formula for Double-Covered Beam

A recurrence formula for the determination of the correction forces for a double-covered beam such as that indicated in figure 1(b), can be obtained in a manner similar to that used for the analysis of the single-covered beam. The equations presented in the previous analysis must be altered, therefore, as indicated below.

(1) In equations (23), (24), and (26), z is replaced by $\frac{h}{2}$ (fig. 5(a)).

(2) Equation (27) can then be written as

$$\delta_i - \delta_{i-1} = \nu'_{i-1}V_{i-1} - \nu'_i V_i \qquad (52)$$

in which

$$\nu' = \frac{1}{2IG}\left\{A_S(h + 2c)\frac{b}{t_c} - \left[A_S(h + 2c) + A_F h\right]\frac{c}{t_w}\right\}$$

(3) Relations (28) are replaced by (see fig. 10)

$$\left.\begin{array}{c} \dfrac{h + 2c}{h}X_i \\[2em] -\dfrac{h + 2c}{h}X_i \end{array}\right\} \qquad (53)$$

and

(4) Equation (7) becomes

$$\bar{\tau}_{w_i} = -\frac{2\bar{\tau}_{c_i}t_{c_i}c}{t_{w_i}h} \qquad (54)$$

From equations (1), (15), and (54), the shear stress in either cover of bay 1 can then be expressed as

$$\left(1 + \frac{2c^2 t_{c_i}}{bh t_{w_i}}\right) \frac{d\bar{\tau}_{c_i}}{dx_i} = \frac{G}{Eb}\left(1 + \frac{2c}{h}\right)\bar{\sigma}_{F_i} - \frac{G}{Eb}\bar{\sigma}_{S_i} \qquad (55)$$

(5) Equations (31) are replaced by

$$\left.\begin{aligned} \bar{\sigma}_{F_i} &= \frac{h + 2c}{hA_{F_i}} X_i \\[2mm] \bar{\sigma}_{S_i} &= -\frac{X_i}{A_{S_i}} \end{aligned}\right\} \qquad (56)$$

(6) From equations (19) and (54) to (56), and (34) and equations similar to equations (32) to (34) and (36), the following expressions for the shear stresses in cover and web which correspond to equations (37) and (38), respectively, are

$$\bar{\tau}_{c_i} = -X_i \frac{K'_i}{t_{c_i}} \frac{\cosh K'_i(L_i - x_i)}{\sinh K'_i L_i} \qquad (57)$$

and

$$\bar{\tau}_{w_i} = 2X_i \frac{K'_i c}{t_{w_i} h} \frac{\cosh K'_i(L_i - x_i)}{\sinh K'_i L_i} \qquad (58)$$

(7) The relationships for the displacement of stringer S at $x_i = 0$ and $x_i = L_i$ that correspond to equations (51) and (52) are:

$$\bar{\delta}_i(0) = \lambda'_i X_i \qquad (59)$$

in which

$$\lambda'_i = \left(1 + \frac{2c^2 t_{c_i}}{bh t_{w_i}}\right)\frac{K'_i b}{Gt_{c_i} \tanh K'_i L_i}$$

and

$$\overline{\delta}_i\left(L_i\right) = \mu'_1 X_1 \tag{60}$$

in which

$$\mu'_1 = \left(1 + \frac{2c^2 t_{c_i}}{bh t_{w_i}}\right) \frac{K'_1 b}{G t_{c_i} \sinh K'_1 L_1}$$

(8) The recurrence formula (45) and the boundary equations (47) to (51), respectively, can then be written for a double-covered box as

$$\mu'_{i-1} X_{i-1} - \left(\lambda'_{i-1} + \lambda'_i\right) X_i + \mu'_i X_{i+1} = \nu'_{i-1} V_{i-1} - \nu'_i V_i \tag{61}$$

$$\mu'_m X_m - \lambda'_m X_{m+1} = \nu'_m V_m \tag{62}$$

$$\lambda'_0 X_0 - \mu'_0 X_1 = \nu'_0 V_0 \tag{63}$$

$$-\left(\lambda'_0 + \lambda'_1\right) X_1 + \mu'_1 X_2 = \nu'_0 V_0 - \nu'_1 V_1 \tag{64}$$

$$\mu'_{m-1} X_{m-1} - \left(\lambda'_{m-1} + \lambda'_m\right) X_m = \nu'_{m-1} V_{m-1} - \nu'_m V_m \tag{65}$$

$$\mu'_m X_m - \left(\lambda'_m + k'\right) X_{m+1} = \nu'_m V_m \tag{66}$$

in which

$$k' = \frac{\lambda'^2_{m+1} - \mu'^2_{m+1}}{\lambda'_{m+1}}$$

when bay $m + 1$ is composed of both skin and stringers, and

$$k' = \frac{I_{m+1}}{hE}\left(\frac{h + 2c}{A_{F_{m+1}}} + \frac{h}{A_{S_{m+1}}}\right)$$

when bay $m + 1$ is composed of stringers only.

Application of Recurrence Formulas and Boundary Equations

Simultaneous equations.- After a beam has been subdivided into the desired number of bays (see fig. 8), an equation in terms of the unknown correction forces X can be written for each bay. Recurrence formula (45) (for single-covered beams) or recurrence formula (61) (for double-covered beams) applies to all bays located at least one bay from a boundary; whereas equations (47) and (48), equations (49) and (50), and equation (51), respectively, apply to fixed-end, unrestrained-end, and partly restrained-end bays of single-covered beams and equations (62) and (63), equations (64) and (65), and equation (66), respectively, apply to fixed-end, unrestrained-end, and partly restrained-end bays of double-covered beams. The group of equations thus obtained constitute a set of simultaneous linear algebraic equations for the determination of the correction forces at each section. A typical set of equations applicable to the beam of figure 8 is given in table 1.

Calculation of stringer stresses.- The total stringer stresses are obtained from the addition of the elementary stringer stresses determined from the formula My/I and the corresponding correction stresses determined by dividing the correction forces at each section considered by the effective stringer areas at that section. If the beam considered is a simplified or "substitute" structure (see, for example, p. 2, reference 2), the elementary stresses should be taken as those of the actual structure. The correction force for the cover stringer S at any section i is obtained directly from the solution of the simultaneous equations; whereas the forces for the web or flange stringers are obtained from relations (28)(for single-covered box) or relations (53) (for double-covered box). The total stresses in stringers F, S, and L, respectively, at section i of a single-covered beam are

$$
\left.
\begin{aligned}
\sigma_{F_i} &= \frac{M_i z_i}{I_i} - \frac{h + c}{h A_{F_i}} X_i \\[2em]
\sigma_{S_i} &= \frac{M_i \left(z_i + c\right)}{I_i} + \frac{X_i}{A_{S_i}} \\[2em]
\sigma_{L_i} &= - \frac{M_i \left(h - z_i\right)}{I_i} + \frac{c}{h A_{L_i}} X_i
\end{aligned}
\right\} \qquad (67)
$$

Similarly, the stresses in stringers F and S in either cover of a double-covered beam are

$$
\left.
\begin{aligned}
\sigma_{F_1} &= \frac{M_1 h}{2I_1} - \frac{h + 2c}{hA_{F_1}} X_1 \\[2ex]
\sigma_{S_1} &= \frac{M_1(h + 2c)}{2I_1} + \frac{X_1}{A_{S_1}}
\end{aligned}
\right\} \tag{68}
$$

Calculation of shear stresses.- The shear stresses in cover and web at any section i are obtained from addition of the elementary shear stresses determined from the formula VQ/It and the correction shear stress τ determined from equations (37) and (38) (for single-covered beams) or equations (57) and (58) (for double-covered beams). The total shear stresses in cover and web, respectively, for a single-covered beam are, therefore,

$$
\tau_{c_1} = \frac{V_1 A_{S_1}\left(z_1 + c\right)}{I_1 t_{c_1}} - \frac{K_1}{t_{c_1}}\left(\frac{X_1 \cosh K_1 L_1 - X_{1+1}}{\sinh K_1 L_1}\right)
$$

and

$$
\tau_{w_1} = \frac{V_1}{I_1 t_{w_1}}\left[A_{S_1}\left(z_1 + c\right) + A_{F_1} z_1\right] + \frac{K_1 c}{t_{w_1} h}\left(\frac{X_1 \cosh K_1 L_1 - X_{1+1}}{\sinh K_1 L_1}\right) \tag{69}
$$

or

$$
\tau_{c_1} = \frac{V_1 A_{S_1}\left(z_1 + c\right)}{I_1 t_{c_1}} - \frac{K_{1-1}}{t_{c_{1-1}}}\left(\frac{X_{1-1} - X_1 \cosh K_{1-1} L_{1-1}}{\sinh K_{1-1} L_{1-1}}\right)
$$

and

$$
\tau_{w_1} = \frac{V_1}{I_1 t_{w_1}}\left[A_{S_1}\left(z_1 + c\right) + A_{F_1} z_1\right] + \frac{K_{1-1} c}{t_{w_{1-1}} h}\left(\frac{X_{1-1} - X_1 \cosh K_{1-1} L_{1-1}}{\sinh K_{1-1} L_{1-1}}\right)
$$

Similarly, for a double-covered beam, the shear stresses in cover and web, respectively, are

$$\tau_{c_i} = \frac{V_i A_{S_i}(h + 2c)}{2I_i t_{c_i}} - \frac{K'_i}{t_{c_i}}\left(\frac{X_i \cosh K'_i L_i - X_{i+1}}{\sinh K'_i L_i}\right)$$

and

$$\tau_{w_i} = \frac{V_i}{2I_i t_{w_i}}\left[A_{S_i}(h + 2c) + A_{F_i}h\right] + \frac{2K'_i c}{t_{w_i} h}\left(\frac{X_i \cosh K'_i L_i - X_{i+1}}{\sinh K'_i L_i}\right) \quad (70)$$

or

$$\tau_{c_i} = \frac{V_i A_{S_i}(h + 2c)}{2I_i t_{c_i}} - \frac{K'_{i-1}}{t_{c_{i-1}}}\left(\frac{X_{i-1} - X_i \cosh K'_{i-1}L_{i-1}}{\sinh K'_{i-1}L_{i-1}}\right)$$

and

$$\tau_{w_i} = \frac{V_i}{2I_i t_{w_i}}\left[A_{S_i}(h + 2c) + A_{F_i}h\right] + \frac{2K'_{i-1}c}{t_{w_{i-1}}h}\left(\frac{X_{i-1} - X_i \cosh K'_{i-1}L_{i-1}}{\sinh K'_{i-1}L_{i-1}}\right)$$

If the corrections are for a substitute structure, the elementary stresses should be taken as those of the actual structure.

NUMERICAL EXAMPLES

Example 1 - Cambered Box Beam with Single Cover

As an illustration of the application of the recurrence formulas and differential equations developed in the present paper, the single-covered cambered box beam of reference 3 is analyzed herein. The beam can be considered as a tip-loaded cantilever box of uniform construction. Pertinent dimensions of both the actual beam and the substitute structure used in reference 3, as well as in the present analysis, are given in figure 11 and in table 2.

Application of differential-equation solution.- Since the beam considered (fig. 11(d)) is tip-loaded and of uniform construction, equations (13) and (14) can be used to determine the shear and stringer stresses, respectively. Since a substitute structure is to be analyzed (fig. 11(c)), only the correction terms of these formulas require consideration. After substitution of the numerical

values given in table 2 into the correction terms of equation (13a), the correction shear stress in the cover becomes

$$\bar{\tau}_c = -21.9 \cosh 0.074(88 - x) \tag{71}$$

in which x is in inches and is measured from the root and $\bar{\tau}_c$ is in pounds per square inch. The elementary shear stress in the part of the cover adjacent to stringer F of the actual structure is

$$\tau'_c = \frac{PQ}{It_c}$$

$$= 9360 \text{ psi} \tag{72}$$

Consequently, the total shear stress in the cover skin immediately adjacent to stringer F is

$$\tau_c = 9360 - 21.9 \cosh 0.074(88 - x) \tag{73}$$

in which τ_c is measured in pounds per square inch. Equations (72) and (73) are plotted in figure 12, along with experimental data obtained for the beam discussed.

The stringer stresses for the substitute structure are determined from equations (14). Substitution of the numerical values given in table 2 into the correction terms of equations (14) yields the following expressions for the correction stresses for stringers F and S:

$$\bar{\sigma}_F = 15.57 \sinh 0.074(88 - x) \tag{74}$$

$$\bar{\sigma}_S = -13.83 \sinh 0.074(88 - x) \tag{75}$$

in which $\bar{\sigma}$ is in pounds per square inch. These corrections are plotted in figure 13.

Although $\bar{\sigma}_F$ is merely added to the corresponding elementary stress to obtain the total stress in stringer F of the actual structure, $\bar{\sigma}_S$ must first be distributed among the three stringers of the actual cover and then each of the three resulting corrections added to the corresponding elementary stringer stresses. In accordance with reference 4, $\bar{\sigma}_S$ is distributed along the developed width of the cross section of the cover as a third-order parabola (fig. 14). Therefore

$$\bar{\sigma} = -\beta + \left(\bar{\sigma}_F + \beta\right)\left(\frac{S}{b_c}\right)^3 \tag{76}$$

with

$\bar{\sigma}$ distributed cover stringer correction stress

β negative of correction stress at center line of cover

S   developed distance from center line of cover

b_c developed length of cover

Since $\bar{\sigma}_S$ is distributed along the developed width

$$\bar{\sigma}_S b_c t_c = \int_0^{b_c} \bar{\sigma} t_c \, ds$$

$$= -b_c t_c \left[\beta - \left(\frac{\sigma_F + \beta}{4}\right)\right]$$

and, consequently

$$\beta = \frac{1}{3}\left(\bar{\sigma}_F - 4\bar{\sigma}_S\right)$$

and, therefore,

$$\bar{\sigma} = \frac{1}{3}\left(4\bar{\sigma}_S - \bar{\sigma}_F\right) + \frac{1}{3}\left(4\bar{\sigma}_F - 4\bar{\sigma}_S\right)\left(\frac{S}{b_c}\right)^3 \tag{77}$$

The correction stress in stringers 1, 2, and 3, respectively, at any distance x from the root is determined from equation (77) by taking $\left(\dfrac{S}{b_c}\right)^3$ equal to 0.364, 0.0787, and 0.00292. (See fig. 14.)

The correction stringer stresses defined by equations (74) and (77) are added to the elementary stresses determined from reference 3, and the resulting total stresses are plotted for the root region in figure 15, together with experimental data.

Application of recurrence-formula solution.- The beam of reference 3 can also be analyzed by the recurrence formula method. Since the beam is single-covered and cantilevered, recurrence formula (45) and boundary equations (47) and (49) are applicable. If the beam is divided into four bays of equal length (fig. 11(e)), m is set equal to 3 in the equations of table 1, and the numerical values of the constants are taken from table 2, the following set of simultaneous equations for the correction forces X_1, X_2, X_3, and X_4 is obtained

X_1	X_2	X_3	X_4	Constants (lb)
-15.55	2.94			0
2.94	-15.55	2.94		0
	2.94	-15.55	2.94	0
		2.94	-7.77	18000

Solution of this set of equations yields:

$X_1 = -18.2$ pounds, $X_2 = -96.4$ pounds, $X_3 = -491.0$ pounds,

and $X_4 = -2500$ pounds.

The correction stringer stresses are obtained from substitution of the numerical values of X into the X-terms of equation (67). For stringers F and S these corrections are indicated by the crosses on figure 13 which fall on the curves obtained from the differential-equation solution. In order to obtain the total stresses in the actual structure the corrections must be considered in a manner similar to that discussed in the preceding section.

The correction shear stresses at each of the four sections considered are obtained directly from the X-terms of equations (69).

These corrections are added to the elementary shear stress determined previously and are indicated by the crosses in figure 12.

Example 2 - Thin-Web Beam

In order to illustrate further the application of the recurrence formulas and differential equations developed, a thin-web beam (fig. 21 of reference 5) is analyzed by use of the formulas and equations for a beam with two covers. A sketch of the beam indicating the method of support, loading, and pertinent dimensions is given in figure 16.

Application of differential-equation solution.- If in equations (21) and (22) (obtained from the solution of the differential equation (20)) the camber c is made equal to the width b (see fig. 6), the panel shown in figure 16 can be analyzed directly. Only the stringer stresses will be determined. From equations (22) and the numerical values given in table 3

$$
\left.
\begin{aligned}
\sigma_F &= 1068 - 22.59x - 27.7 \sinh 0.0674(47.25 - x) \\
\\
\sigma_S &= 2135 - 45.18x + 7.914 \sinh 0.0674(47.25 - x)
\end{aligned}
\right\} \quad (78)
$$

in which x is measured from the root in inches and σ is in pounds per square inch.

The stresses obtained with equations (78) are plotted in figure 17 and as indicated coincide with the stresses determined in reference 5.

Application of recurrence-formula solution.- The stresses in the thin-web beam can also be determined by application of recurrence formula (61) and boundary equations (62) and (64). Division of the beam into three bays of equal length (fig. 16) and application of equations (64), (61), and (62), respectively, yields the following set of simultaneous equations for the correction forces X:

X_1	X_2	X_3	Constants
$-2\lambda'$	μ'		0
μ'	$-2\lambda'$	μ'	0
	μ'	$-\lambda'$	$\nu'P$

Substitution of the numerical values given in table 3 yields

X_1	X_2	X_3	Constants (lb)
-26.8	8.29		0
8.29	-26.8	8.29	0
	8.29	-13.4	-1094

and, consequently

$$X_1 = 10.96 \text{ pounds}$$

$$X_2 = 35.4 \text{ pounds}$$

$$X_3 = 103.6 \text{ pounds}$$

Substitution of these correction forces and the numerical values given in table 3 into equations (68) and (70), respectively, yields the stringer and shear stresses in the panel at sections 1, 2, and 3. The stringer stresses obtained are plotted in figure 17.

ACCURACY OF ANALYSIS

The results of the numerical examples presented indicate that the recurrence formulas and differential equations developed in the present paper are suitable for the analyses of the stringer and shear stresses in cambered box beams. The plots given in figure 13 show that the correction stringer stresses determined from the recurrence-formula solution are identical with those found by the differential-equation solution, as they should be because the two solutions are fundamentally identical. The curves for the total stresses in the stringers in the root region (fig. 15) indicate that the differential-equation solution (and consequently the recurrence-formula solution) yields results in good agreement with the experimental data of reference 3. The plots given in figure 12 for the shear stresses in the skin of the part of the cover adjacent to the flange stringer also indicate that the results of the recurrence-formula solution and differential-equation solution are in good agreement with the experimentally determined stresses. It should be noted that for both stringer stress and shear stress the elementary solution yields results that are in poor agreement with the experimental data.

The stresses in the stringers of the thin-web beam (example 2) of reference 5 determined by the recurrence-formula solution and differential-equation solution are identical with those found by the method of analysis of reference 5 and are plotted in figure 17.

CONCLUSIONS

Recurrence formulas and differential equations are developed for the stress analysis of cambered box beams having either one or two covers. The recurrence formulas in conjunction with the boundary equations presented can be used for the determination of stringer and shear stresses in laterally loaded cambered box beams restrained at both ends, unrestrained at both ends, or restrained at one end and unrestrained at the other end. The differential-equation solutions are applicable to cantilevered cambered box beams that are tip-loaded and of uniform construction.

Comparisons given between the methods of analysis of the present paper and experimental data and another method of analysis indicate that the formulas and equations developed are adequate for the stress analysis of cambered box beams.

Langley Memorial Aeronautical Laboratory
    National Advisory Committee for Aeronautics
        Langley Field, Va., July 15, 1947

REFERENCES

1. Kuhn, Paul: Stress Analysis of Beams with Shear Deformation of the Flanges. NACA Rep. No. 608, 1937.

2. Kuhn, Paul: Approximate Stress Analysis of Multistringer Beams with Shear Deformation of the Flanges. NACA Rep. No. 636, 1938.

3. Peterson, James P.: Shear-Lag Tests of a Box Beam with a Highly Cambered Cover in Tension. NACA ARR No. L5F27b, 1945.

4. Kuhn, Paul: A Procedure for the Shear-Lag Analysis of Box Beams. NACA ARR, Jan. 1943.

5. Ebner, H., and Köller, H.: Über den Kraftverlauf in längs- und querversteiften Scheiben. Luftfahrtforschung, Bd. 15, Lfg. 10/11, Oct. 10, 1938, pp. 527-542.

TABLE 1.- SCHEME OF EQUATIONS FOR CANTILEVER CAMBERED BOX BEAM

$$X_1 \qquad X_2 \qquad X_3 \qquad X_{m-1} \qquad X_m \qquad X_{m+1} \qquad \text{Constants}$$

$$-(\lambda_0 + \lambda_1) \qquad \mu_1 \qquad\qquad\qquad\qquad\qquad \nu_0 V_0 - \nu_1 V_1$$

$$\mu_1 \qquad -(\lambda_1 + \lambda_2) \qquad \mu_2 \qquad\qquad\qquad \nu_1 V_1 - \nu_2 V_2$$

$$\mu_{m-1} \quad -(\lambda_{m-1} + \lambda_m) \quad \mu_m \quad \nu_{m-1}V_{m-1} - \nu_m V_m$$

$$\mu_m \qquad -\lambda_m \qquad \nu_m V_m$$

TABLE 2.- NUMERICAL VALUES FOR EXAMPLE 1

P (lbs)	c (in.)	b (in.)	h (in.)	t_w (in.)	t_c (in.)	L (in.)
3000	3.38	9.01	4.85	0.0782	0.0251	88

A_F (in.^{2})	A_S (in.^{2})	A_L (in.^{2})	I (in.^{4})	z (in.)	G (psi)	E (psi)	L_1 (in.)	b_c (in.)
0.810	0.537	3.98	42.3	3.28	4×10^6	10.4×10^6	22	16.63

TABLE 3.- NUMERICAL VALUES FOR EXAMPLE 2

P (lbs)	$c = b = \frac{h}{2}$ (in.)	$t_w = t_c$ (in.)	L (in.)	A_F (in.^{2})	A_S (in.^{2})	I (in.^{4})	G (psi)	E (psi)	L_1 (in.)
2205	9.84	0.0315	47.25	0.620	1.085	960	4×10^6	10.4×10^6	15.75
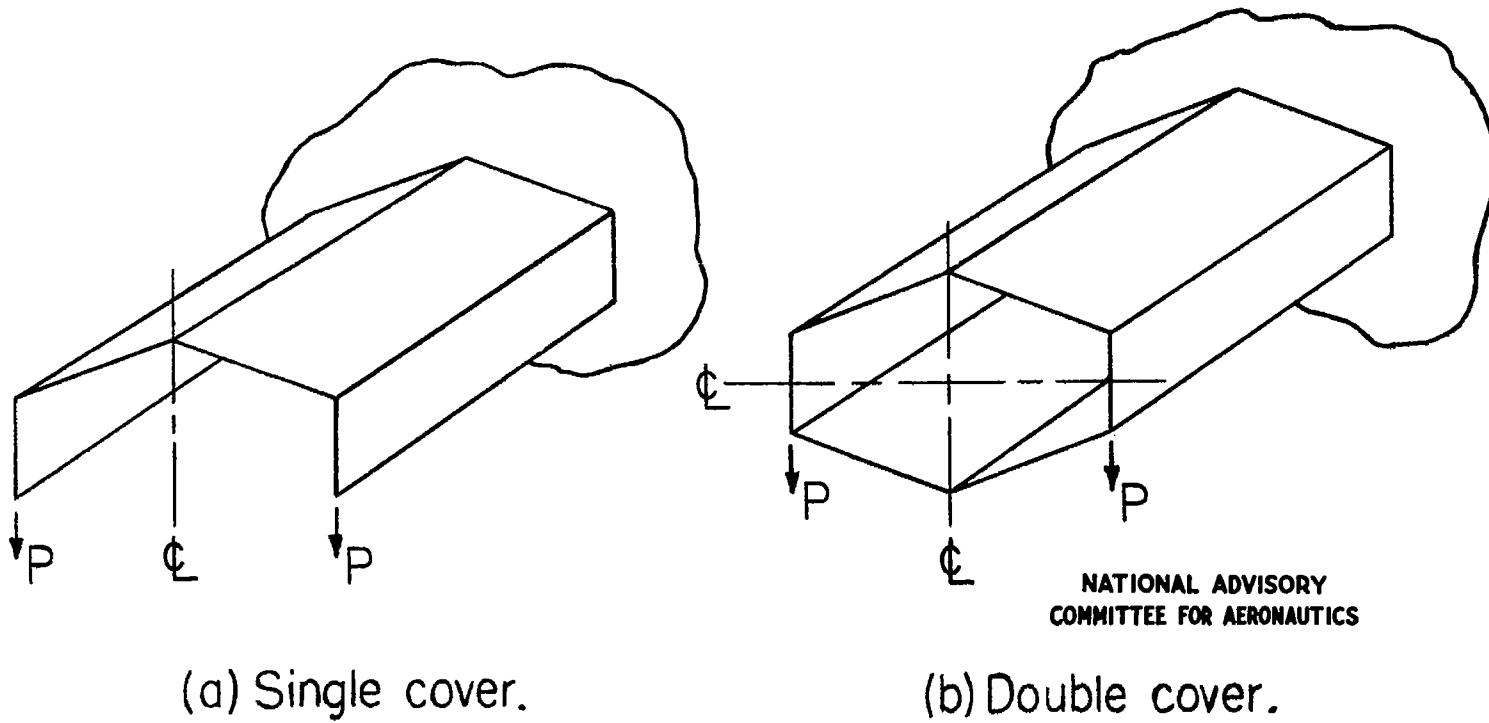
(a) Single cover.

(b) Double cover.

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Figure 1.- Cambered box beams.

(a) Beam loaded at several sections.

(b) Stringer displacements due to cover shear.

(c) Total cover stringer displacements due to shear.

(d) Stringer displacements due to bending.
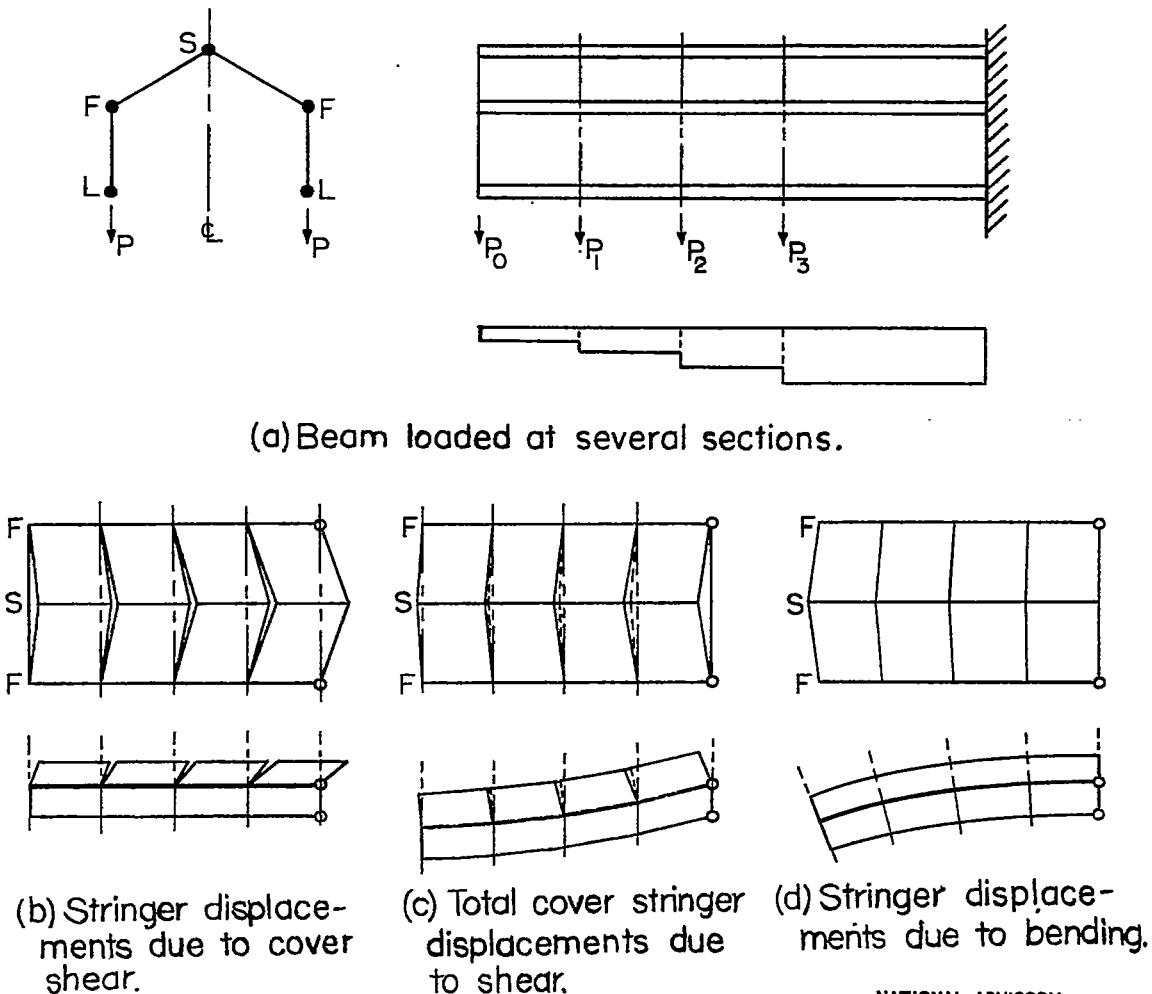
Figure 2.- Cover stringer discontinuities.
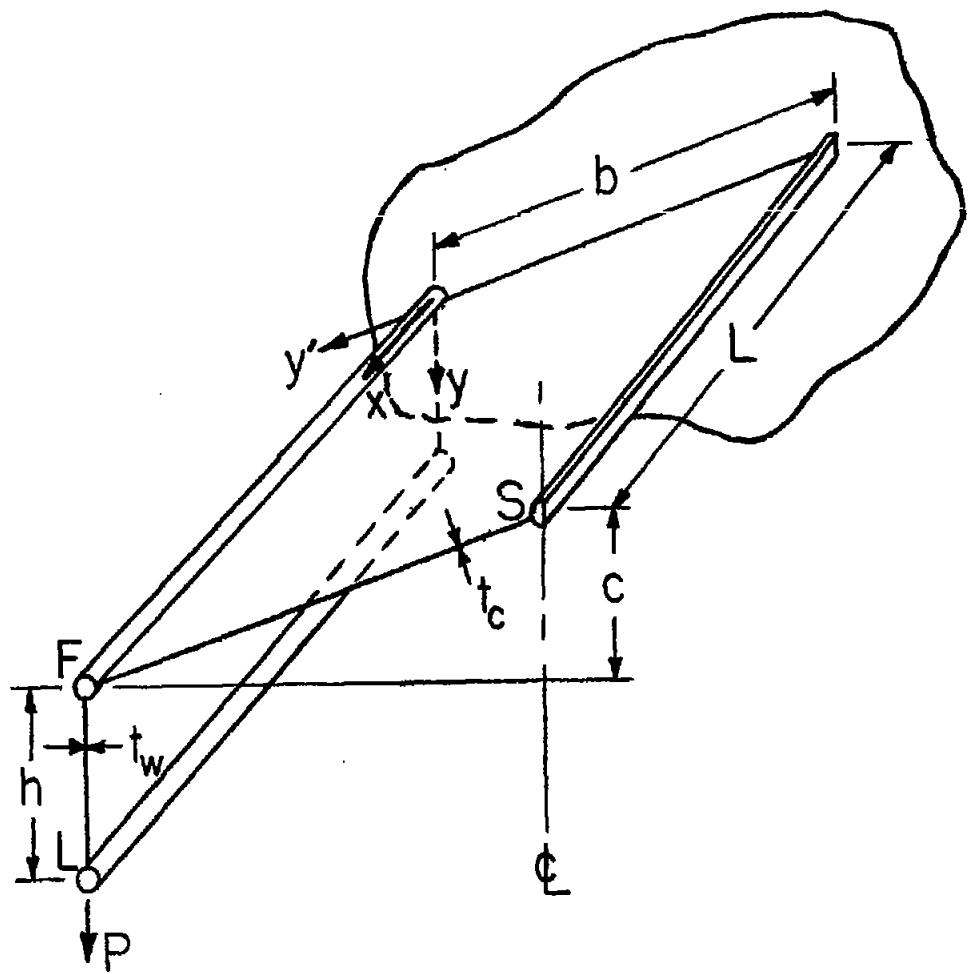
(a) Single covered box beam.
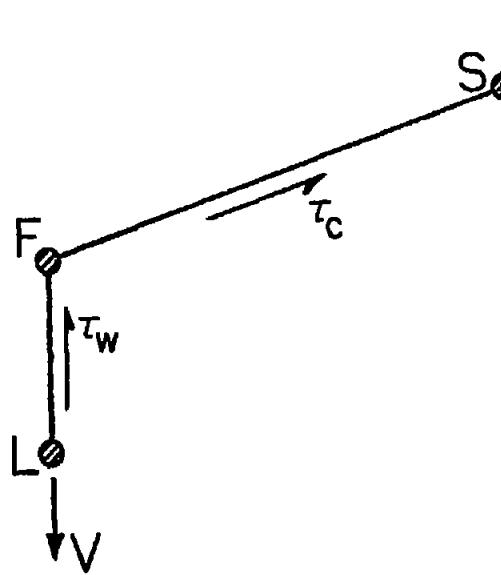
(b) Shear stresses.

Figure 3.- Notation for tip-loaded single-covered box beam.
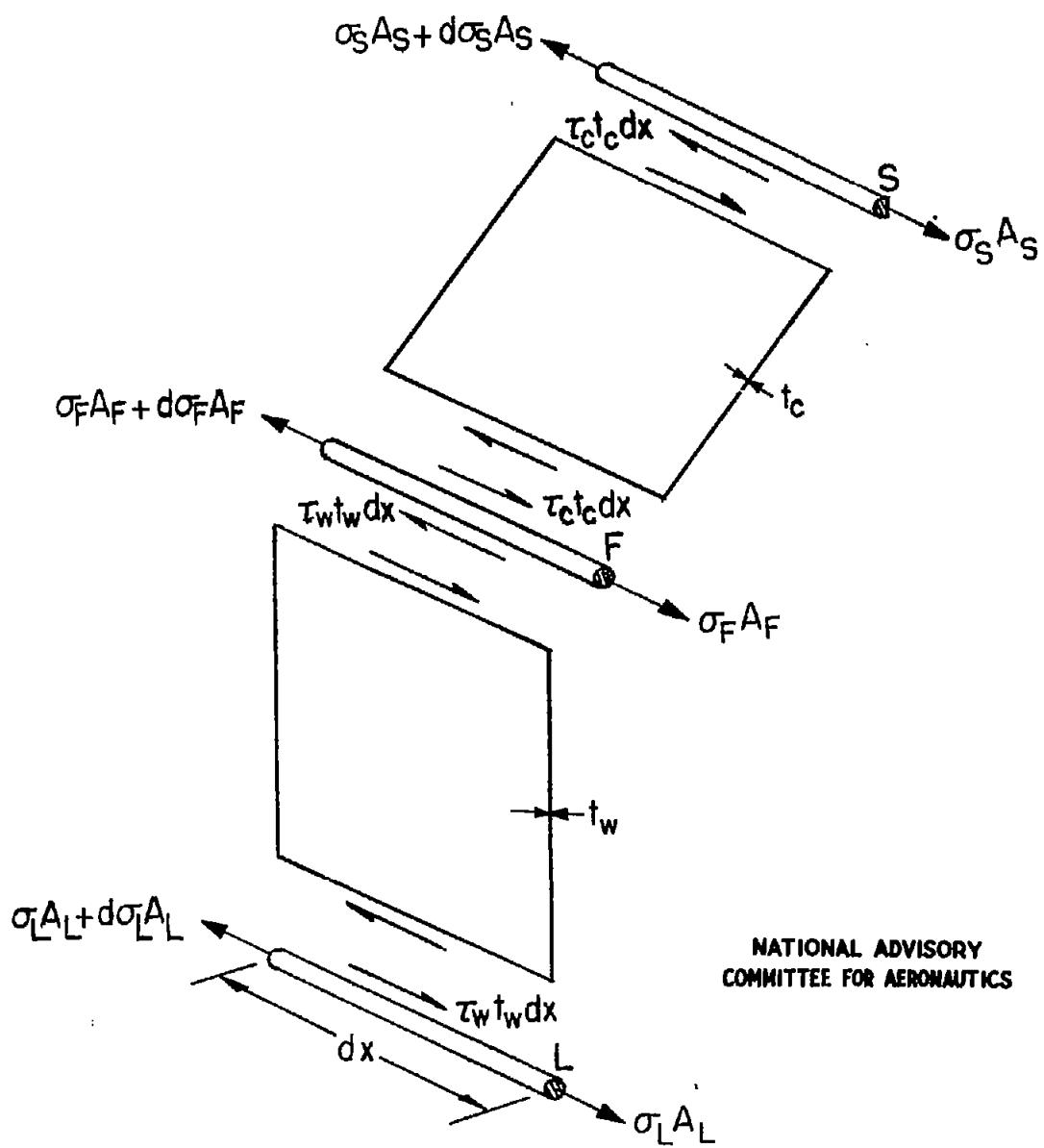
Figure 4.- Free body diagram of beam.
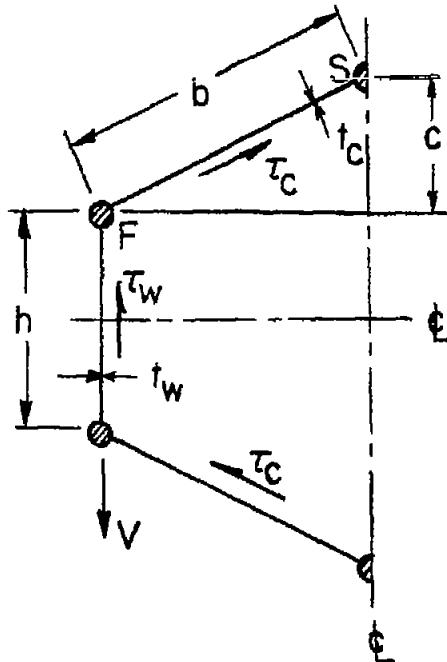
(a) Shear stresses.
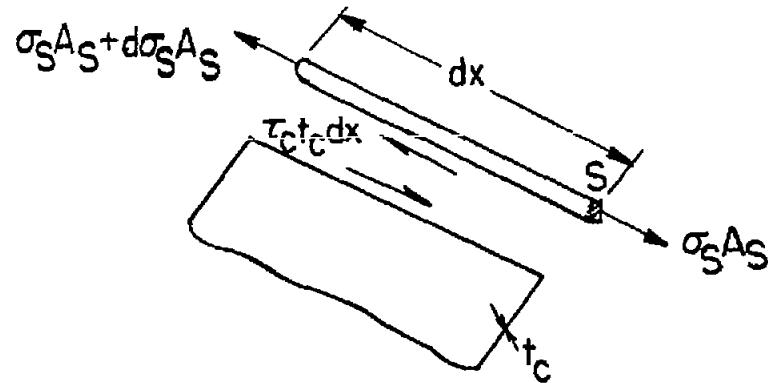
(b) Cover stringer.
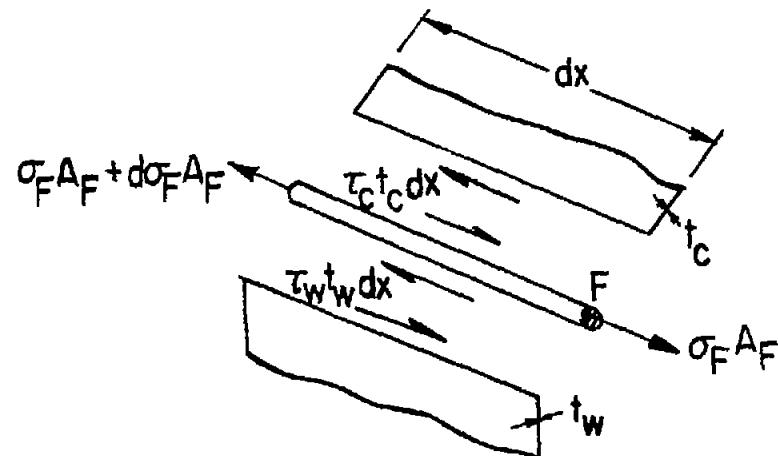
(c) Web stringer.

Figure 5.- Double covered box beam.
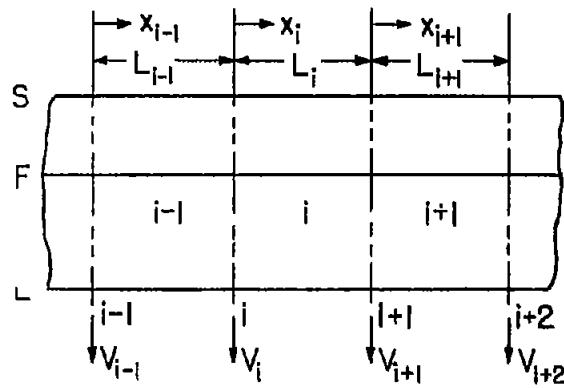
(a) Convention for numbering bays and stations.
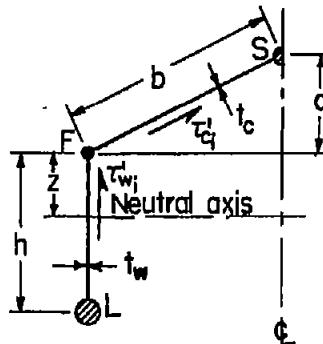
(b) Elementary shear stresses.
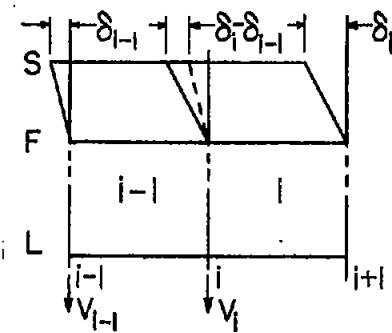
(c) Cover stringer displacements.
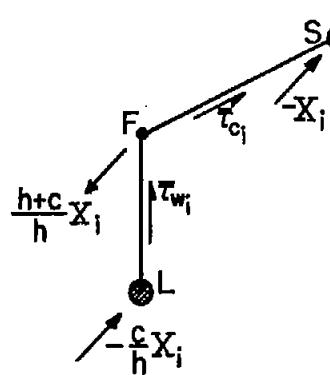
(d) Correction forces and shear stresses.

Figure 6.- Notation for single covered box beams.
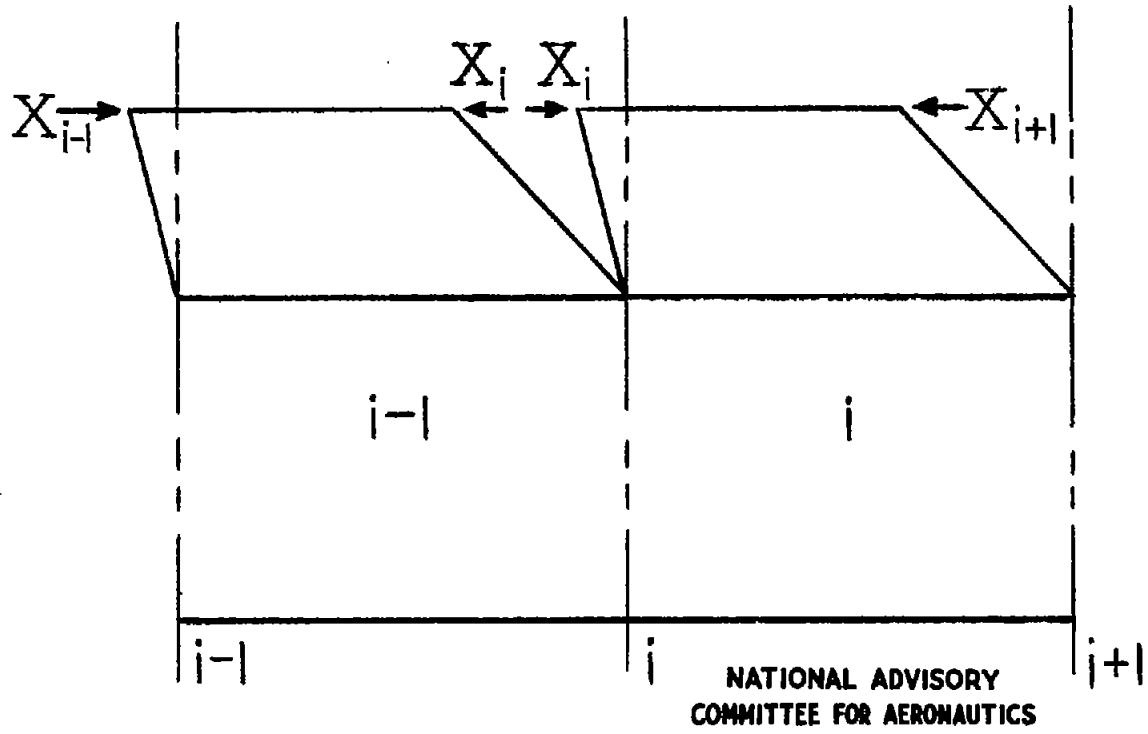
Figure 7.-Application of correction forces.
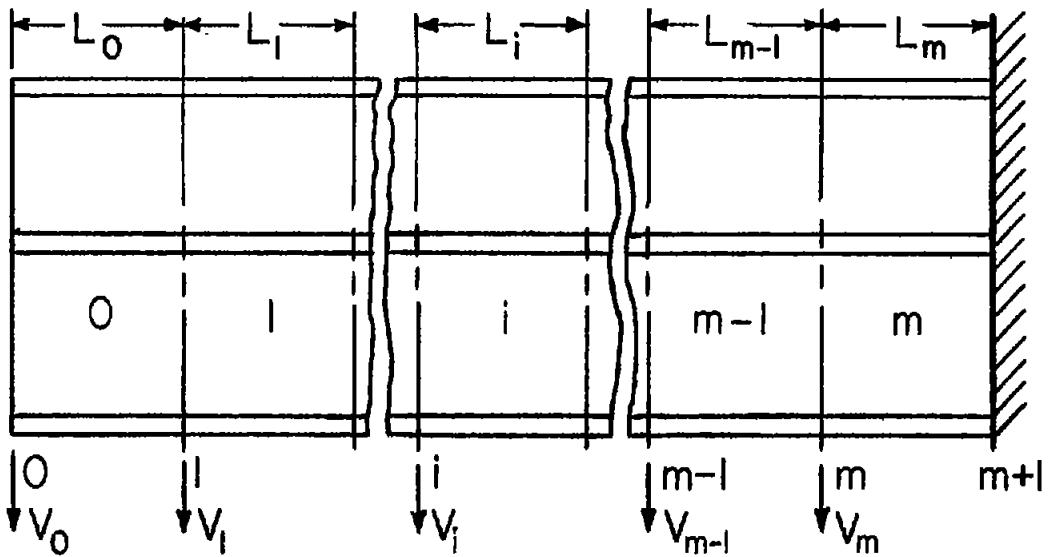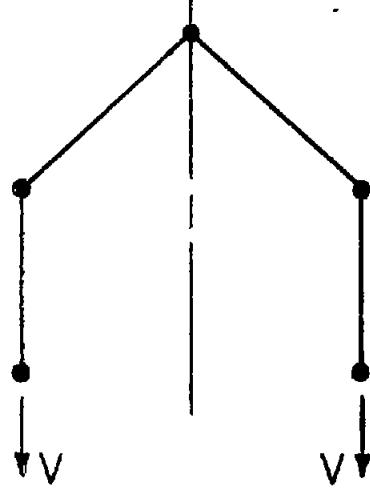
Figure 8.- Beam with fixed end.
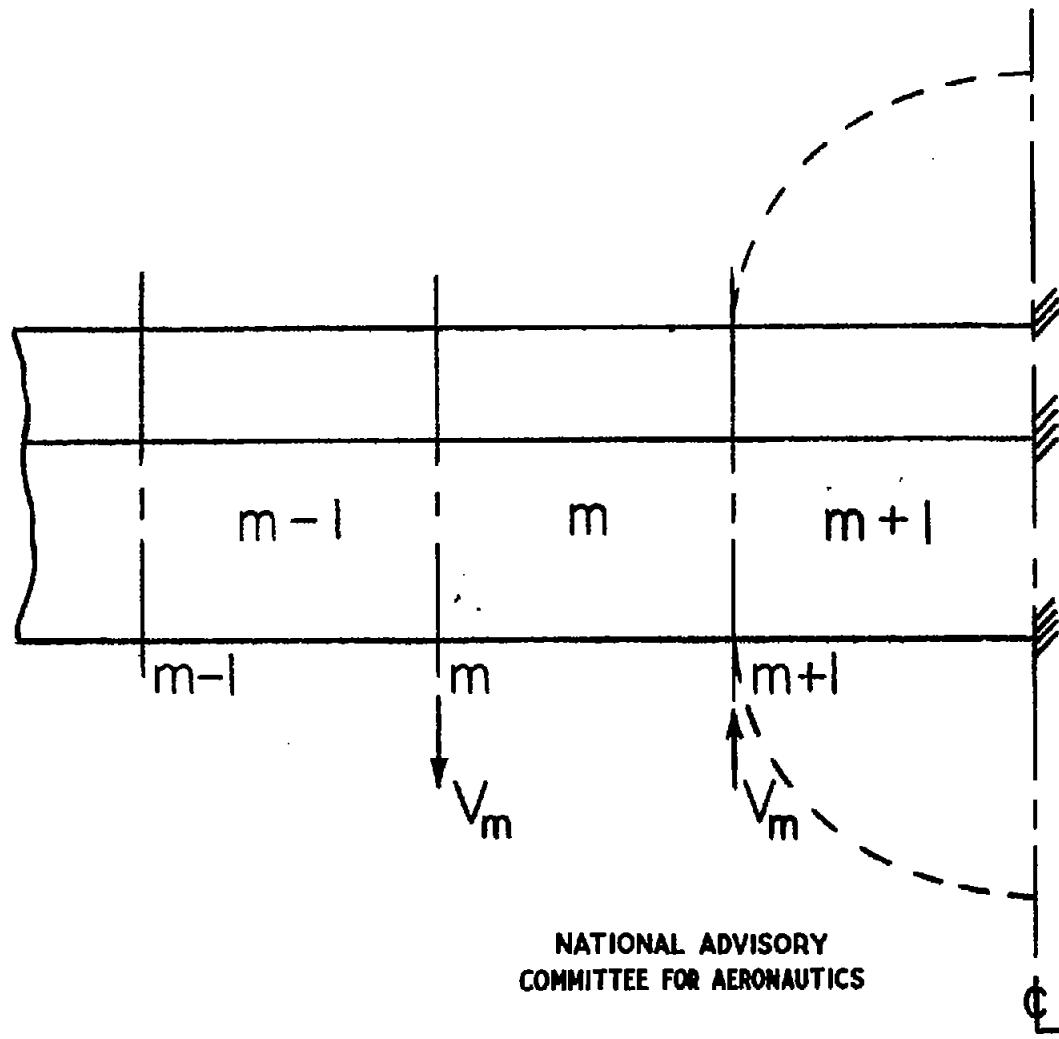
Figure 9.- Beam with carry-through bay.

$$S$$

$$\overline{\tau}_{c_i} \quad -X_i$$

$$F$$

$$\frac{h+2c}{h}X_i \qquad \overline{\tau}_{w_i}$$

$$-\frac{h+2c}{h}X_i$$

$$\overline{\tau}_{c_i}$$

$$X_i$$
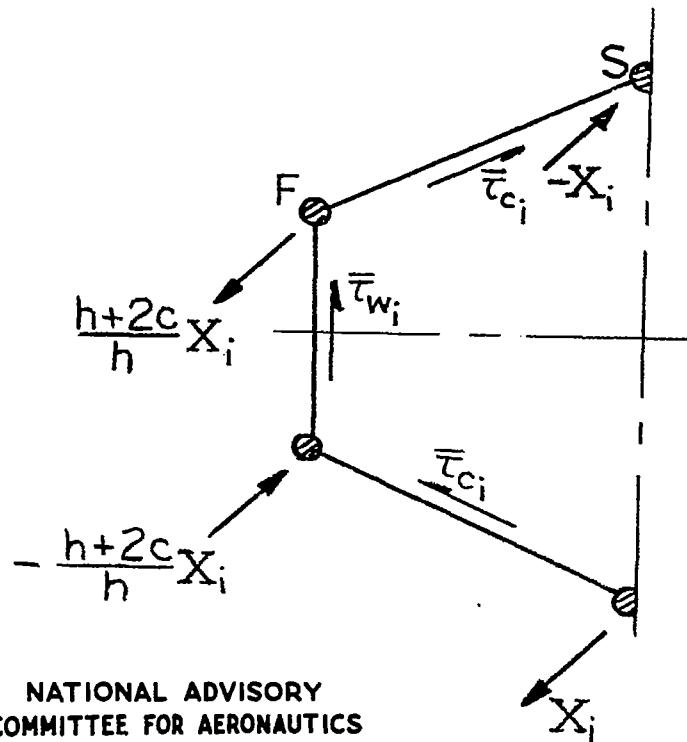
Figure 10.-Correction forces for
double covered box beam.
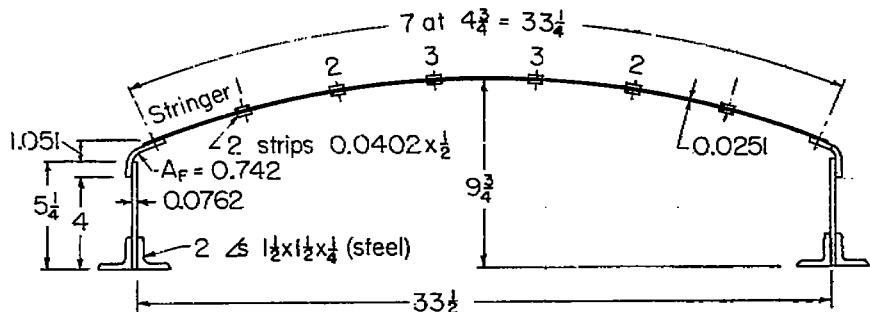
(a) Cross section of test beam.
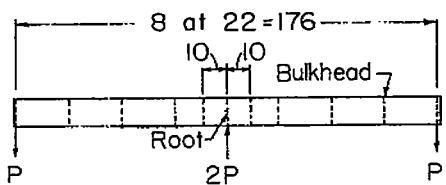
(b) Side elevation of test beam.
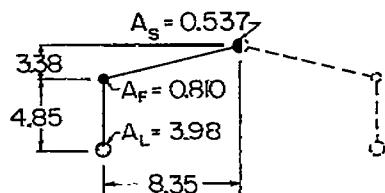
(c) Substitute single stringer beam.
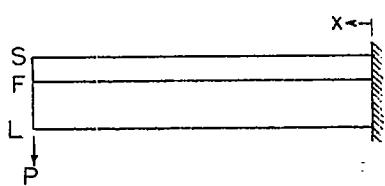
(d) Side elevation of substitute beam.
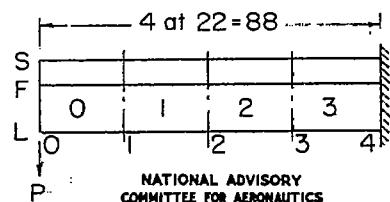
(e) Notation for substitute beam.

Figure 11.- Details of test beam and substitute single stringer beam   (example 1).
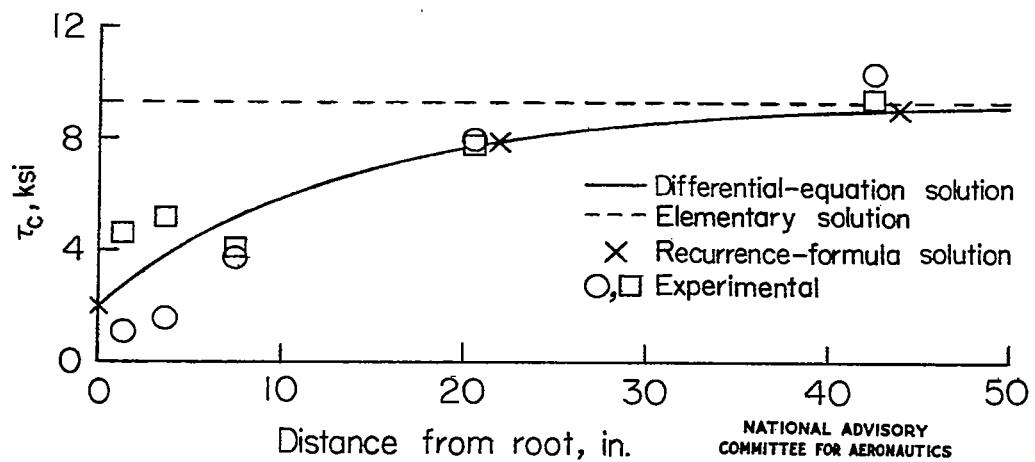
Figure 12.- Cover shear stress (example 1).
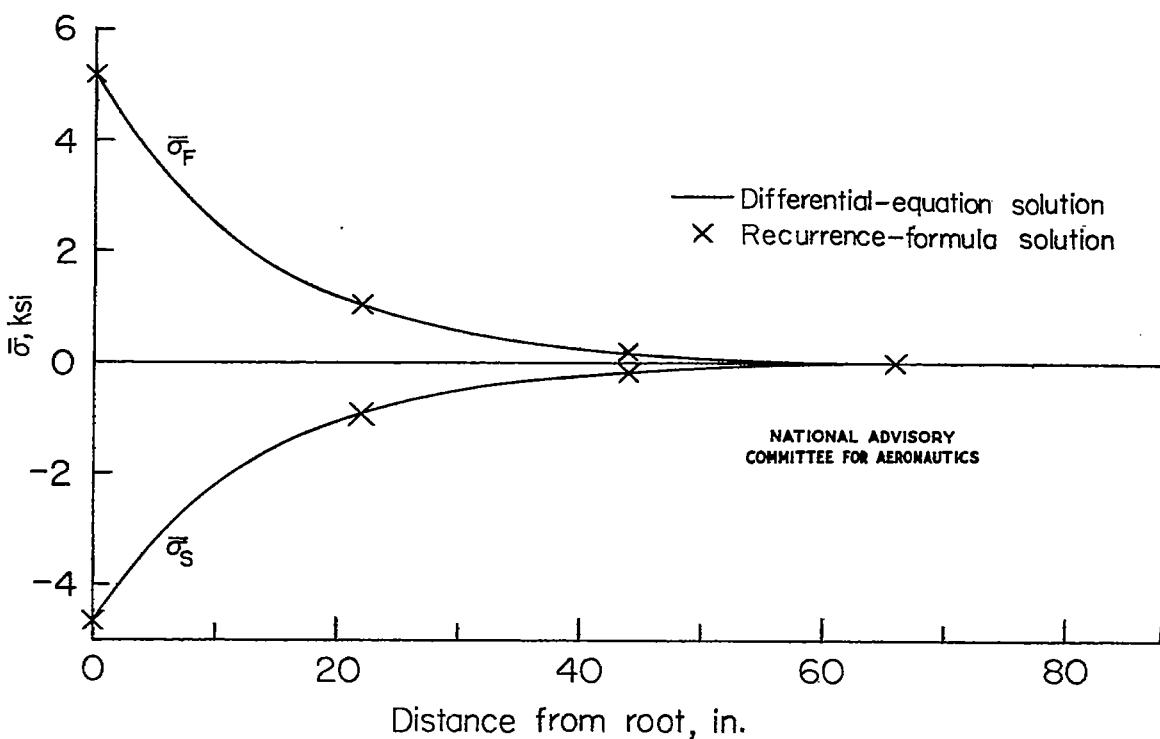

Figure 13.- Stringer correction stresses (example 1).
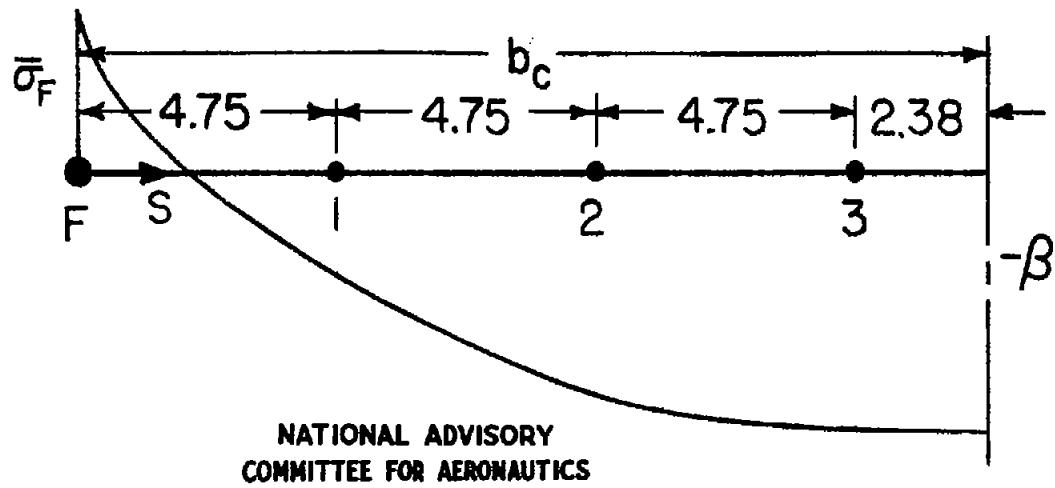
Figure 14.- Distribution of correction stresses (example 1).
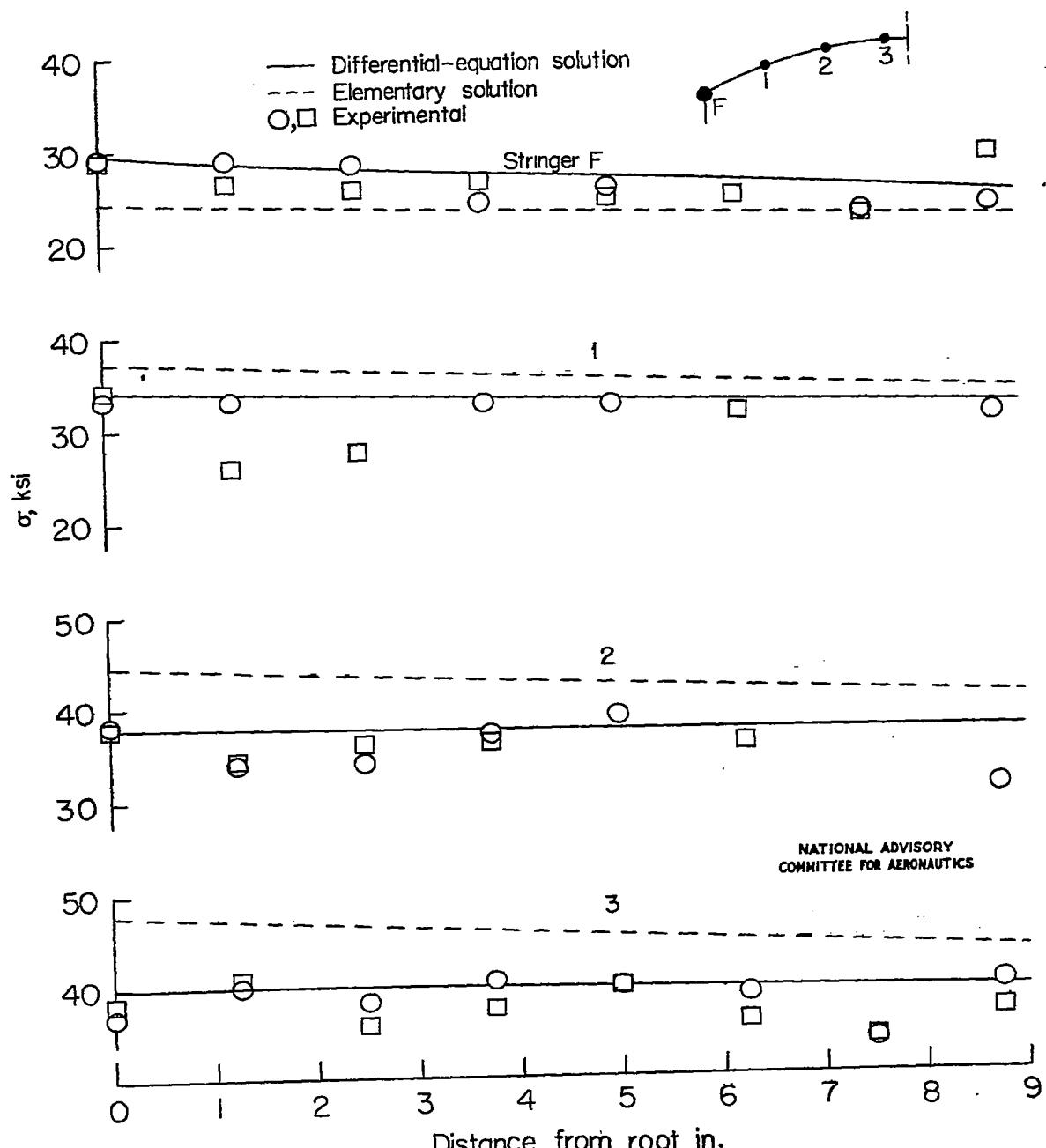
Figure 15.- Cover stringer stresses (example 1).
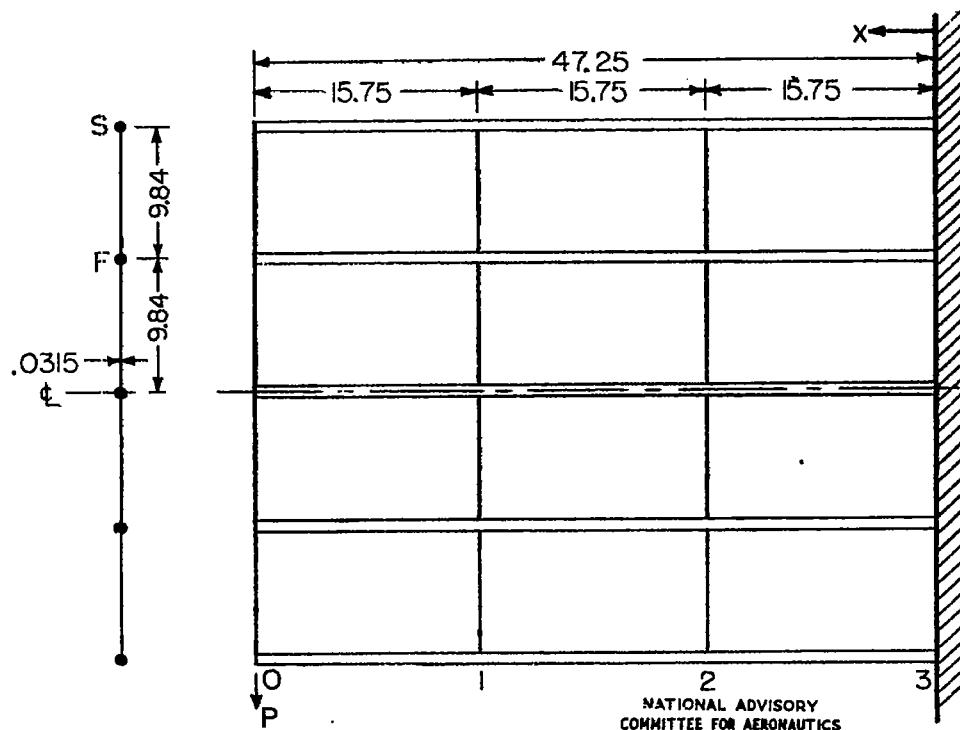
Figure 16.- Thin-web beam (example 2).

Distance from root, in.

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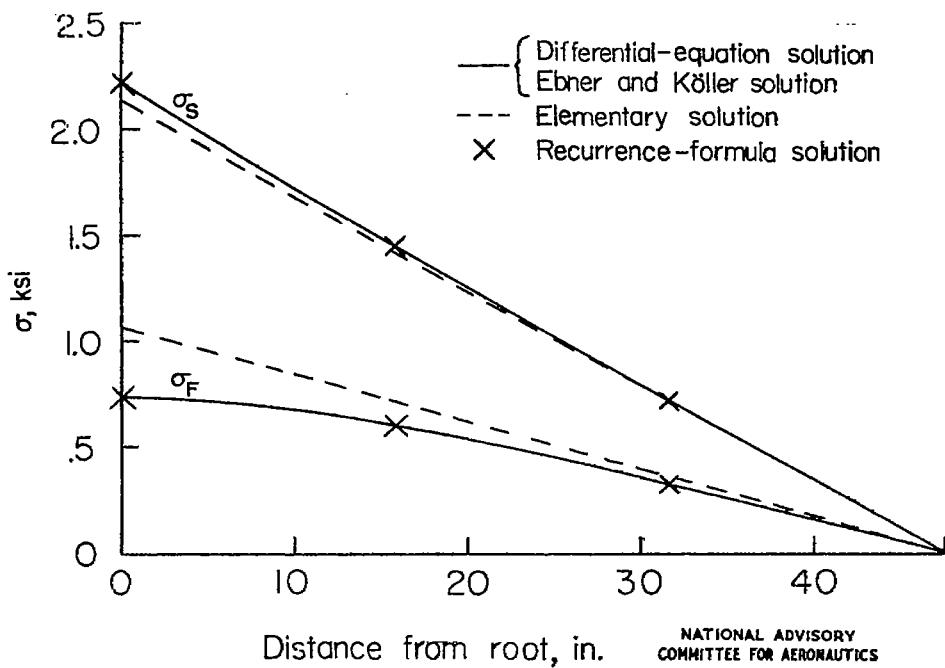
Figure 17.- Stringer stresses in thin-web beam (example 2).